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PLASTIC FLOWS OF A MISES MEDIUM WITH HELICAL SYMMETRY†

S. I. SENASHOV

Krasnoyarsk

e-mail: senashov@kgtei.kts.ru

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Series of exact solutions of the Mises plasticity equations which possess helical symmetry are presented. They can be used to analyse the stress–strain state of circular rods and tubes acted upon by an internal pressure, an axial force and a torque. © 2004 Elsevier Ltd. All rights reserved.

In constructing exact solutions of problems in the theory of plasticity with the Mises yield condition in the axisymmetric and three-dimensional cases, it is usually necessary to act in the inverse mode, i.e. first to construct an exact solution and then to attempt to match an actual physical problem to it. Nevertheless, this approach enables one to solve many important practical problems of mechanics: to make estimates of limiting loads, to construct stress fields, etc. [1–5]. The number of solutions obtained in this manner remains extremely limited [5] and this remark from some 40 years ago still holds up to the present time. The small number of solutions does not permit one to study the structure of the equations thoroughly, to prove existence and uniqueness theorems and to test numerical calculations. The problem of the construction of exact solutions is therefore urgent at the present time.

The group analysis of differential equations is one of the most powerful techniques for solving problems using inverse methods. Its use immediately enables one to construct new classes of exact solutions of the plasticity equations with the Mises yield condition [6]. The book by Annin *et al.* [7] was the first work in this area.

We will consider the equations of ideal plasticity with the Mises yield condition in the tree-dimensional case, written in the cylindrical system of coordinates r, z and θ. We will introduced a new variable by the formula $\xi = z + k\theta$ and we shall assume that all of the components of the velocity vector u, υ and w and the hydrostatic pressure p depend solely on the two variables r and ξ. The equations in terms of stresses have the form

$$\begin{aligned}
&\frac{\partial S_r}{\partial r} + \frac{k}{r}\frac{\partial S_{r\theta}}{\partial \xi} + \frac{\partial S_{rz}}{\partial \xi} + \frac{2S_r - S_\theta}{r} = \frac{\partial p}{\partial r}\\
&\frac{\partial S_{r\theta}}{\partial r} + \frac{k}{r}\frac{\partial S_\theta}{\partial \xi} + \frac{\partial S_{\theta z}}{\partial \xi} + \frac{2S_{r\theta}}{r} = \frac{k\partial p}{r\partial \xi}\\
&\frac{\partial S_{rz}}{\partial r} + \frac{k}{r}\frac{\partial S_\theta}{\partial \xi} + \frac{\partial S_z}{\partial \xi} + \frac{S_{rz}}{r} = \frac{\partial p}{\partial \xi}\\
&S_r^2 + S_\theta^2 + S_z^2 + 2(S_{r\theta}^2 + S_{rz}^2 + S_{\theta z}^2) = 2k_s^2, \quad S_r + S_\theta + S_z = 0\\
&S_r = \lambda u_r, \quad S_\theta = \lambda r^{-1}(k\upsilon_\xi + u), \quad S_z = \lambda \omega_\xi\\
&2S_{r\theta} = \lambda(r^{-1}ku_\xi + r(r^{-1}\upsilon)_r), \quad 2S_{rz} = \lambda(\upsilon_\xi + \omega_r), \quad 2S_{\theta z} = \lambda(r^{-1}k\omega_\xi + \upsilon_\xi)
\end{aligned} \tag{1}$$

Here, S_r, S_θ, S_z, $S_{r\theta}$, S_{rz}, $S_{\theta z}$, are the components of the stress tensor and k_s is the yield point.

System (1) describes the plastic flow of a substance under the condition of helical symmetry. When $k = 0$ and $\upsilon = 0$, these equations become the equations of axisymmetric deformation and possess a higher

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degree of symmetry compared with them. This enables us not only to construct generalized axisymmetric solutions [6] but also solutions which do not have axisymmetric analogues.

We will seek a solution of system (1) in the form

$$u = u(r)\sin\xi, \quad v = v(r)\cos\xi, \quad \omega = \omega(r)\cos\xi, \quad p = p(r, \xi)$$

Suppose $S_{r\theta} = S_{rz} = 0$. Then, the remaining components of the stress tensor deviator depend solely on the single variable r. If $p = p(r)$, the second and the third equations of system (1) are identically satisfied and the first equation serves to determine $p(r)$.

In this case, we obtain the system of ordinary differential equations

$$ku + rv' - v = 0, \quad \omega' + u = 0, \quad ru' - kv + u - \omega = 0$$

for determining the functions u, v and w from system (1), which reduces to Bessel's equation

$$r^2u'' + ru' + (r^2 + k^2 - 1)u = 0$$

Solving this equation, we obtain

$$u = C_1J_\nu + C_2Y_\nu, \quad \nu^2 = 1 - k^2$$

When $|k| \le 1$, the function u takes real values. If, however, $|k| > 1$, it is possible to use the integral representation of a Bessel function

$$J_\nu = \frac{2(z/2)^\nu}{\Gamma(\nu + 1/2)\pi^{1/2}} \int_0^{\pi/2} \cos(z\cos t)\sin^{2\nu} t dt$$

taking account of just the real part.

Finally, the solution has the form (when $C_2 = 0$)

$$u = AJ_\nu\sin\xi, \quad v = -Ark\cos\xi\int_0^r J_\nu r^{-1}dr, \quad \omega = -A\cos\xi\int_0^r J_\nu dr$$

where J_ν is the Bessel function of imaginary argument which satisfies the condition $J_\nu(0) = 0$ for all $\nu > 0$ and A is an arbitrary constant.

In the case of non-zero components of the stress tensor deviator and the hydrostatic pressure, we have

$$\begin{gathered} S_r = -(1+f)S_\theta, \quad S_\theta = k_s(1 + f + f^2 + \varphi^2)^{-1/2}, \quad S_z = fS_\theta \\ p = S_r - \int_a^r (2+f)S_\theta r^{-1}dr, \quad S_{\theta z} = \varphi S_\theta; \quad f = \frac{S_z}{S_\theta}, \quad \varphi = \frac{S_{\theta z}}{S_\theta} \end{gathered} \tag{2}$$

This solution can be interpreted, in particular when $A > 0$, as the plastic flow of a circular tube acted upon by an internal pressure p, an axial force N and a torque M for which

$$\sigma_r|_{r=a} = -p, \quad \sigma_r|_{r=b} = 0, \quad N = 2\pi\int_a^b \sigma_z r dr, \quad M = 2\pi\int_a^b S_{\theta z} r dr$$

where a and b are the internal and external radii of the tube. When $k = 0$ and $\nu = 0$, this solution was constructed by Hill [5].

The system of equations (1) admits of a Lie algebra of operators with the basis

$$A_1 = \partial_\xi, \quad A_2 = \partial_\omega, \quad A_3 = \partial_p, \quad A_4 = u\partial_u + v\partial_v + \omega\partial_\omega, \quad A_5 = r\partial_v$$

The optimal system of subalgebras for the Lie algebra L_5 has the form

$$\theta_1: A_2 \pm A_5, A_4, A_2, A_5$$

$$\theta_2: (A_2 \pm A_5, A_4), (A_2, A_4), (A_5, A_4), (A_2, A_5)$$

Here we have taken into account that the operators A_1, A_3 generate the centre of the Lie algebra L_5.
The optimal system of subalgebras which has been constructed enables one to enumerate all of the different invariant solutions of Eqs (1), apart from symmetry transformations.
We now construct a solution which is invariant with respect to the subalgebra $A_4 - A_1$. One must seek this solution in the form

$$u = u_0(r)e^{\xi}, \quad \upsilon = \upsilon_0(r)e^{\xi}, \quad \omega = \omega_0(r)e^{\xi}, \quad p = p_0(r)$$

The zero subscripts are omitted everywhere below.
It follows from Eqs (1) that

$$S_{r\theta} = C_1/r^2, \quad S_{rz} = C_2/r$$

If the arbitrary constants C_1 and C_2 are equal to zero, we obtain the system of ordinary differential equations

$$ku' + r\upsilon' - \upsilon = 0, \quad \omega' + u = 0, \quad ru' + k\upsilon + u + \omega = 0 \tag{3}$$

It reduces to Bessel's equation, the solution of which has the form

$$u = AJ_\nu(r), \quad \nu^2 = 1 + k^2 \tag{4}$$

Assuming that the function u is bounded when $r = 0$ (otherwise, we add a MacDonald function to expression (4)), we obtain

$$u = AJ_\nu, \quad \omega = -\int_0^r u dr, \quad k\upsilon = -r\omega - ru' - u$$

where J_ν is the Bessel function of imaginary argument which satisfies the condition $J_\nu(0) = 0$ for all $\nu > 0$ and A is an arbitrary constant. The stressed state is described by formulae (2). The mechanical interpretation is the same as in the previous case.
If we put $k = 0, v = 0$, then $M = 0$ and we obtain the axisymmetric solution [8], which described the plastic flow of a cylinder with a stress-free lateral surface.
The stressed state (2) is obtained if we seek the solution of Eqs (1) in the form (subject to the condition $S_{r\theta} = S_{rz} = 0$)

$$u = u_0(r)\operatorname{sh}\xi, \quad \upsilon = \upsilon_0(r)\operatorname{ch}\xi, \quad w = w_0(r)\operatorname{ch}\xi, \quad p = p_0(r)$$

This leads to system (3), and the solution has precisely the same mechanical interpretation as was mentioned above.
We will seek an invariant solution in the subgroup $A_1 + \alpha A_2 + \alpha A_5$ in the form

$$u = f(r), \quad \omega - \alpha\xi = \varphi(r), \quad \upsilon - \gamma\xi r = \psi(r), \quad p = p(r), \quad \gamma = \beta/\alpha$$

We substitute these relations into system (1).
From the incompressibility equation we have (C_1 and C_2 are constants)

$$f = C_1 r + C_2 r^{-1}, \quad C_1 = (\alpha + \gamma k)/2$$

Suppose $\varphi = \psi = 0$; then $S_{r\theta} = S_{rz} = 0$, and the remaining components of the stress tensor deviator have the form

$$S_r = \lambda(C_1 - C_2 r^{-2}),\quad S_\theta = \lambda(k\gamma + C_1 + C_2 r^{-2}),\quad S_z = \alpha\lambda,\quad 2S_{\theta z} = \lambda(\alpha k r^{-1} + \gamma r)$$

$$p = S_r + \int_a^r \lambda(-k\gamma - 2C_2 r^{-2})dr$$

$$2k_s^2\lambda^{-2}r^4 = \left\lfloor (C_2 - C_1 r^2)^2 + [(k\gamma + C_1)r^2 + C_2]^2 + (\gamma r^3 + akr)/2 + a^2 r^4 \right\rfloor$$

This solution describes the limiting state of a tube sited upon by a constant internal pressure p_0, an axial force N and a torque M. When $k = 0, v = 0$ it becomes the axisymmetric solution [3] or a solution [1] which describes the compression of the plastic layer by the coaxial cylindrical surfaces.

It is well known [9] that helical surfaces $z + k\theta = \text{const}$ in a twisted plastic rod possess a number of remarkable properties: they serve as the boundary between the rigid and plastic domains and are the most probable fracture surfaces. We will now construct a solution which describes plastic flow with such surfaces.

We will seek a solution of system of equations (1) in the form

$$u = 0,\quad v = -r\varphi(\xi),\quad \omega = k\varphi(\xi),\quad p = p(r) \tag{5}$$

Substituting expressions (5) into system (1), we obtain the exact solution

$$\begin{aligned} &u = 0,\quad v = -r\varphi(\xi),\quad \omega = k\varphi(\xi),\quad \rho = \rho(r)\\ &\sigma_r = S_r - p = 2\chi k_s \operatorname{arctg}\frac{r}{k} + c\\ &\sigma_\theta = S_\theta - p = -2\chi k_s \frac{rk}{r^2 + k^2} + \sigma_r\\ &\sigma_z = S_z - p = 2\xi k_s \frac{rk}{r^2 + k^2} + \sigma_r\\ &S_{r\theta} = S_{rz} = 0,\quad S_{\theta z} = \chi k_s \frac{r^2 - k^2}{r^2 + k^2};\quad \chi = \operatorname{sign}\varphi'(\xi) \end{aligned} \tag{6}$$

where ψ is an arbitrary smooth function and c is an arbitrary constant.

This solution can be used to describe the plastic flow of a circular rod of radius R acted upon by a tensile force and a torque. Suppose the lateral surface of the rod stress-free and that we require to satisfy the condition $S_{\theta z} = 0$. We obtain from this that $k = \pm R$. The velocity $w = R\varphi(R\theta)$ is specified at the end $z = 0$. Since the function φ is continuous, we obtain that, here, it is a 2π-periodic function, and this means that it has at least one point where $\varphi' = 0$, that is, a helical surface ξ_0 exists such that $\varphi'(\xi_0) = 0$. It follows from formula (6) that, in accordance with the reasoning in [9], which argues that the surfaces $z + k\theta = \text{const}$ can be separated into rigid and plastic regions, a rigid domain arises along this surface in the rod. These surfaces are also the most probable fracture surfaces.

It follows from formula (6) that, by choosing the function φ, it is possible to specify different plastic flows and, in particular, it is possible to describe the technological process of the extrusion of a material between two helical surfaces $\xi = C_1$, $\xi = C_2$. This process could serve as a basis, for example, for the manufacture of drills.

We will now point one further exact solution of Eqs (1)

$$\begin{aligned} u &= \xi(cr^{-1} - (a + kb)r)\\ v &= \xi br + k(a + kb)r\ln r + ckr^{-1}/2 + c_1 r\\ \omega &= \xi^2 a + (a + kb)r^2/2 - c\ln r + c_2 \end{aligned}$$

where a, b, c, c_1, c_2 are arbitrary constants. The components of the strain rates are given by the expressions

$$e_r = -\xi(cr^{-2} + (a+kb)), \quad e_\theta = 2kb\xi + \xi(cr^{-1} + (a+kb)), \quad e_z = 2\xi a$$

$$e_{r\theta} = e_{rz} = 0, \quad e_{\theta z} = 2ak\xi r^{-1} + 2kr\xi$$

In this case, only a single rigid domain exists and it is given by the equation $\xi = 0$. The above-mentioned solution can be used to describe the plastic flow of a circular tube which has been cut along the generatrix.

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